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On Quasi-Interpolation with Non-uniformly Distributed Centers on Domains and Manifolds

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The paper studies quasi-interpolation by scaled shifts of a smooth and rapidly decaying function. The centers are images of a smooth mapping of the $h\mathbf{Z}^n$-lattice in $\mathbf{R}^s$, $s \geqslant n$, and the scaling parameters are proportional to h. We show that for a large class of generating functions the quasi-interpolants provide high order approximations up to some prescribed accuracy. Although in general the approximants do not converge as h tends to zero, the remaining saturation error is negligible in numerical computations if a scalar parameter is suitably chosen. The lack of convergence is compensated for by a greater flexibility in the choice of generating functions used in numerical methods for solving operator equations. © 2001 Academic Press

1. INTRODUCTION

In this paper we continue the study of an approximation method, which was recently proposed in [7, 8] and is mainly directed to the numerical solution of multidimensional integro-differential and other operator equations. In its simplest form, analysed in [9, 10], the method is based on the quasi-interpolation from shift-invariant spaces generated by sufficiently smooth and rapidly decaying functions and is characterised by an accurate approximation up to any prescribed tolerance. However, in general the approximation spaces do not reproduce polynomials and the quasi-interpolants do not converge. For that reason such processes were called *approximate approximations.*

The lack of convergence, which by a proper choice of a scalar parameter can be made not perceptible in numerical computations, is offset by a greater flexibility in the choice of approximating functions. So it is possible to construct multivariate approximation formulas, which are easy to implement and have additionally the property that pseudo-differential operations can be effectively performed. This allows to construct new classes of semi-analytic cubature formulas of integral operators of mathematical physics

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and to create effective numerical algorithms for solving boundary value problems for differential and integral equations (cf. [6, 8, 10]).

The present paper is devoted to an extension of this method to the case that the data points are not uniformly distributed and to the approximation of functions on a manifold. We study the approximation properties of quasi-interpolants of the form

$$\mathscr{D}^{-n/2} \sum u(\mathbf{x_m})\, \eta\left(\frac{\mathbf{x}-\mathbf{x_m}}{\sqrt{\mathscr{D}}\, V_{\mathbf{m}}}\right), \qquad \mathbf{x} \in \Omega, \tag{1.1}$$

where the function u has compact support on some domain $\Omega \subset \mathbf{R}^n$ or n-dimensional manifold $\Omega \subset \mathbf{R}^s$. The generating function η has to be sufficiently smooth and rapidly decaying and to satisfy the moment conditions

$$\int_{\mathbf{R}^n} \eta(\mathbf{y})\, d\mathbf{y} = 1, \qquad \int_{\mathbf{R}^n} \mathbf{y}^{\boldsymbol{\alpha}} \eta(\mathbf{y})\, d\mathbf{y} = 0, \tag{1.2}$$

for all multiindices $\boldsymbol{\alpha}$ with $1 \leqslant |\boldsymbol{\alpha}|_1 < N$. If Ω is an n-dimensional manifold then the function η is assumed to be radial. The data points or centers $\{\mathbf{x_m}\} \in \Omega$ are the images of a lattice of width h under smooth parametrization of Ω. We require that the scaling $V_{\mathbf{m}}$ is proportional to h and connected with the mesh near the data point $\mathbf{x_m}$. Due to the fast decay of η the summation in (1.1) extends only about centers in a neighbourhood B of $\mathbf{x}$ with diameter proportional to h. We show that for any $\varepsilon > 0$ there exists $\mathscr{D} > 0$ such that at any point $\mathbf{x} \in \Omega$

$$\left| u(\mathbf{x}) - \mathscr{D}^{-n/2} \sum_{\mathbf{x_m} \in B} u(\mathbf{x_m})\, \eta\left(\frac{\mathbf{x}-\mathbf{x_m}}{\sqrt{\mathscr{D}}\, V_{\mathbf{m}}}\right)\right|$$

$$\leqslant c(\sqrt{\mathscr{D}}\, h)^N \, \|u\|_{C^N(\bar{\Omega})} + \varepsilon \sum_{k=0}^{N-1} c_k (\sqrt{\mathscr{D}}\, h)^k, \tag{1.3}$$

where c does not depend on u, h, and $\mathscr{D}$ and the numbers c_k can be determined from the values of the partial derivatives of u at the point $\mathbf{x}$. Additionally, for any given η the functional dependence of ε upon the parameter $\mathscr{D}$ is explicitly known. Therefore, in numerical computations the parameter $\mathscr{D}$ can be chosen such that ε is less than any prescribed accuracy. Then the last term in (1.3), which we call saturation error because it does not converge to zero, can be neglected and the approximation process and numerical algorithms using it behave like Nth order approximation processes. The use of those approximation processes in practical applications is quite reasonable, since there is always a natural bound for the accuracy, determined for example by the tolerance of the input data or the computer system.

Estimates in different norms of the above mentioned type are proved in [9, 10] for quasi-interpolation with uniformly distributed centers, i.e., $\mathbf{x_m} = h\mathbf{m}$, $\mathbf{m} \in \mathbf{Z}^n$, and $V_{\mathbf{m}} = h$, corresponding to the stationary case of principal shift-invariant spaces. As mentioned before, in practical applications the generating function in (1.1) is chosen such that the action of a given pseudodifferential operator on η results in an effectively computable or even an analytic expression. So one obtains effective approximations of pseudodifferential operators which are in many cases subjected to estimates of the type (1.3). Note that for pseudodifferential operators of negative order, for example the convolution with the fundamental solution of a partial differential equation, even the saturation error tends to zero (see [10]).

However, in many practical problems it is impossible to use approximants with uniformly distributed centers. For example, to compute the values of integral operators on domains by using this approach the density will be approximated by a linear combination of smooth functions which is given on the whole space. An accurate approximation in integral norms together with reasonable numerical costs can be achieved only with certain mesh refinement towards the boundary. It is interesting that for the classes of generating functions under consideration wavelet-based methods can be developed resulting in specially adapted meshes for a given density function (cf. [11]). A direct approach to mesh refinement was introduced in [5]. Here simple approximation formulas are studied for functions given on some polyhedral domain which are based on multilevel quasi-interpolation with piecewise uniformly distributed centers refined towards the boundary. It is shown that these approximants provide estimates of the type (1.3) in integral norms, such that high order semi-analytic cubature formulas for the classical potentials over polyhedral domains are available. Therefore the results obtained here allow us also to treat the important case of integral operators over domains with curved boundary.

There is already substantial literature on multivariate quasi-interpolation if the nodes have a regular distribution. The case of irregularly distributed or even scattered data points is one of the main research topics, especially in the context of the radial basis function theory. Interpolation by translates of a given radial basis function has become a well-studied method to approximate functions sampled at scattered points (see, e.g., [13] and the references therein). Quasi-interpolation with quite nonregular distributed centers was studied in [4], where one has to abstain from the convolution form of the approximant. Moreover, the quasi-interpolation formula requires to build for each center $\mathbf{x_m}$ a special linear combination $\eta_{\mathbf{m}}$ of shifts of the basis function. Note that the convolutional form of the quasi-interpolant (1.1) is possible due to our restriction to centers resulting from a sufficiently smooth mapping of a lattice.

The paper is organized as follows. In Section 2 we briefly review results for the quasi-interpolation with gridded centers needed in the proof of estimate (1.3). There we give also an analytic formula to obtain generating functions satisfying the moment conditions (1.2) with large N. In Section 3 we prove the estimate (1.3) for a special case of the quasi-interpolant (1.1) involving explicitly the parametrization of Ω. Finally, in Section 4 we extend this result to the more general formula (1.1) and give an example of quasi-interpolation on a surface in $\mathbf{R}^3$ with $N=4$.

2. RESULTS FOR GRIDDED CENTERS

In this section we consider quasi-interpolation from stationary shift-invariant spaces with a continuous generating function η possessing the decay

$$|\eta(\mathbf{x})| \leqslant A_K(1+|\mathbf{x}|^2)^{-K/2}, \qquad \mathbf{x}\in\mathbf{R}^n, \tag{2.1}$$

for some number $K>N+n$, $N\geqslant 1$ is some given integer, and constant A_K. Here $|\mathbf{x}| := |\mathbf{x}|_2 = \sqrt{\mathbf{x}\cdot\mathbf{x}}$ denotes the Euclidean norm in $\mathbf{R}^n$.

For given positive parameter $\mathscr{D}$ and multiindex $\boldsymbol{\alpha}=(\alpha_1, ..., \alpha_n)\in\mathbf{Z}^n_{\geqslant 0}$ we introduce the functions

$$\sigma_{\boldsymbol{\alpha}}(\mathbf{x}, \mathscr{D}) := \mathscr{D}^{-n/2} \sum_{\mathbf{m}\in\mathbf{Z}^n} \left(\frac{\mathbf{x}-\mathbf{m}^{\boldsymbol{\alpha}}}{\sqrt{\mathscr{D}}}\right) \eta\left(\frac{\mathbf{x}-\mathbf{m}}{\sqrt{\mathscr{D}}}\right),$$

$$\rho_{\boldsymbol{\alpha}}(\mathbf{x}-\mathscr{D}) := \mathscr{D}^{-n/2} \sum_{\mathbf{m}\in\mathbf{Z}^n} \left|\left(\frac{\mathbf{x}-\mathbf{m}}{\sqrt{\mathscr{D}}}\right)^{\boldsymbol{\alpha}} \eta\left(\frac{\mathbf{x}-\mathbf{m}}{\sqrt{\mathscr{D}}}\right)\right|.$$

We use the notations $\mathbf{x}^{\boldsymbol{\alpha}} = x_1^{\alpha_1}\cdots x_n^{\alpha_n}$, $[\boldsymbol{\alpha}] := |\boldsymbol{\alpha}|_1 = \alpha_1+\cdots+\alpha_n$, $\boldsymbol{\alpha}! = \alpha_1!\cdots\alpha_n!$ and

$$\partial^{\boldsymbol{\alpha}} u(\mathbf{x}) = \frac{\partial^{[\boldsymbol{\alpha}]}}{\partial x_1^{\alpha_1}\cdots\partial x_n^{\alpha_n}}\, u(\mathbf{x}).$$

LEMMA 2.1. *For any* $\mathscr{D}_0>0$ *and* $\boldsymbol{\alpha}$, $0\leqslant[\boldsymbol{\alpha}]\leqslant N$, *there exist constants* $c_{\boldsymbol{\alpha}}$ *such that for all* $\mathscr{D}\geqslant\mathscr{D}_0$

$$\|\sigma_{\boldsymbol{\alpha}}(\cdot, \mathscr{D})\|_{L^\infty(\mathbf{R}^n)} \leqslant \|\rho_{\boldsymbol{\alpha}}(\cdot, \mathscr{D})\|_{L^\infty(\mathbf{R}^n)} \leqslant c_{\boldsymbol{\alpha}}.$$

LEMMA 2.2. *Suppose that the function* $u \in L^\infty(\mathbf{R}^n)$ *is N-times continuously differentiable in the closed ball centered at* $\mathbf{x} \in \mathbf{R}^n$ *with radius* $\kappa > 0$, $u \in C^N(B(\mathbf{x}, \kappa))$.
Then the semi-discrete convolution

$$\mathcal{M}_h u(\mathbf{x}) := \mathcal{D}^{-n/2} \sum_{\mathbf{m} \in \mathbf{Z}^n} u(h\mathbf{m})\, \eta\left(\frac{\mathbf{x} - h\mathbf{m}}{\sqrt{\mathcal{D}}\, h}\right) \tag{2.2}$$

can be represented in the form

$$\mathcal{M}_h u(\mathbf{x}) = \sum_{[\boldsymbol{\alpha}=0]}^{N-1} \frac{\partial^{\boldsymbol{\alpha}} u(\mathbf{x})}{\boldsymbol{\alpha}!} (-\sqrt{\mathcal{D}}\, h)^{[\boldsymbol{\alpha}]}\, \sigma_{\boldsymbol{\alpha}}(\mathbf{x}/h, \mathcal{D}) + R_N(\mathbf{x}, \sqrt{\mathcal{D}}\, h),$$

where

$$|R_N(\mathbf{x}, \sqrt{\mathcal{D}}\, h)| \leqslant (\sqrt{\mathcal{D}}\, h)^N \sum_{[\boldsymbol{\alpha}]=N} \frac{\|\partial^{\boldsymbol{\alpha}} u\|_{C(B(\mathbf{x}, \kappa))}}{\boldsymbol{\alpha}!}\, \rho_{\boldsymbol{\alpha}}(\mathbf{x}/h, \mathcal{D}) + o((\sqrt{\mathcal{D}}\, h)^N).$$

With the abbreviation $e_{\boldsymbol{\lambda}}(\mathbf{x}) := e^{2\pi i \mathbf{x} \cdot \boldsymbol{\lambda}}$ the Fourier transform is defined as

$$\mathcal{F}_\varphi(\boldsymbol{\lambda}) = \int_{\mathbf{R}^n} \varphi(\mathbf{x})\, e_{\boldsymbol{\lambda}}(-\mathbf{x})\, d\mathbf{x}.$$

The application of Poisson's summation formula yields

LEMMA 2.3. *If for the given* $\mathcal{D} > 0$ *and* $\boldsymbol{\alpha}$ *the sequence*

$$\{\partial^{\boldsymbol{\alpha}} \mathcal{F}\eta(\sqrt{\mathcal{D}}\,\cdot)\} \in l_1(\mathbf{Z}^n)$$

then

$$\sigma_{\boldsymbol{\alpha}}(\mathbf{x}, \mathcal{D}) = \left(\frac{i}{2\pi}\right)^{[\boldsymbol{\alpha}]} \sum_{\boldsymbol{\nu} \in \mathbf{Z}^n} \partial^{\boldsymbol{\alpha}} \mathcal{F}\eta(\sqrt{\mathcal{D}}\, \boldsymbol{\nu})\, e_{\boldsymbol{\nu}}(\mathbf{x}). \tag{2.3}$$

Thus the semi-discrete convolution (2.2) represents a quasi-interpolant approximating locally sufficiently smooth functions u with the order $\mathcal{O}((\sqrt{\mathcal{D}}\, h)^N)$ if $\sigma_\alpha(\mathbf{x}, \mathcal{D}) = \delta_{[\boldsymbol{\alpha}]\,0}$ for all $\mathbf{x} \in \mathbf{R}^n$. These equalities imply the well-known Strang-Fix conditions

$$\partial^{\boldsymbol{\alpha}} \mathcal{F}\eta(\mathbf{0}) = \delta_{[\boldsymbol{\alpha}]\,0}, \qquad \partial^{\boldsymbol{\alpha}} \mathcal{F}\eta(\sqrt{\mathcal{D}}\, \boldsymbol{\nu}) = 0, \qquad \boldsymbol{\nu} \in \mathbf{Z}^n \setminus \{\mathbf{0}\}, \qquad 0 \leqslant [\boldsymbol{\alpha}] \leqslant N-1.$$

The main idea of approximate approximations is to use generating functions η for which the functions $\sigma_{\boldsymbol{\alpha}}(\mathbf{x}, \mathcal{D})$ can be made arbitrarily close to $\delta_{[\boldsymbol{\alpha}]\,0}$ by choosing appropriate values of $\mathcal{D}$. It turns out that there exists a large class of those functions with very useful analytic properties, which of course have to satisfy the moment condition (1.2). This follows together with some construction from the following two lemmas.

LEMMA 2.4. *Suppose that η is differentiable of order M with the smallest integer $M > n/2$ and satisfies together with all derivatives $\partial^{\boldsymbol{\beta}}\eta$, $[\boldsymbol{\beta}] \leqslant M$, the decay condition* (2.1). *Then for any $\varepsilon > 0$ there exists $\mathscr{D} > 0$ such that for all $\boldsymbol{\alpha}$, $0 \leqslant [\boldsymbol{\alpha}] \leqslant N$,*

$$\sum_{\mathbf{v} \in \mathbf{Z}^n \setminus \{\mathbf{0}\}} |\partial^{\boldsymbol{\alpha}} \mathscr{F}\eta(\sqrt{\mathscr{D}}\, \mathbf{v})| < \varepsilon. \tag{2.4}$$

Proof. In view of Lemma 2.1 and the decay (2.1) the 1-periodic function $\partial^{\boldsymbol{\beta}}\sigma_{\boldsymbol{\alpha}}(\mathbf{x}, \mathscr{D})$ belongs to $L^\infty(\mathbf{R}^n)$ and satisfies

$$\|\partial^{\boldsymbol{\beta}}\sigma_{\boldsymbol{\alpha}}(\cdot, \mathscr{D})\|_{L^\infty(\mathbf{R}^n)} \leqslant c_{\boldsymbol{\beta}} \mathscr{D}^{-M/2} \qquad \text{for all} \quad [\boldsymbol{\beta}] = M$$

with some constants $c_{\boldsymbol{\beta}}$ not depending on $\mathscr{D}$. Thus by Parceval's equality we obtain

$$\sum_{[\boldsymbol{\beta}]=M} \sum_{\mathbf{v} \in \mathbf{Z}^n} (2\pi \mathbf{v})^{2\boldsymbol{\beta}}\, |\partial^{\boldsymbol{\alpha}} \mathscr{F}\eta(\sqrt{\mathscr{D}}\, \mathbf{v})|^2 = \sum_{[\boldsymbol{\beta}]=M} \|\partial^{\boldsymbol{\beta}}\sigma_{\boldsymbol{\alpha}}(\cdot, \mathscr{D})\|^2_{L^2([0,1]^n)}$$
$$\leqslant c\mathscr{D}^{-M}.$$

Since

$$\sum_{[\boldsymbol{\beta}]=M} (2\pi \mathbf{v})^{2\boldsymbol{\beta}} \geqslant \gamma\, |\mathbf{v}|^{2M}$$

with some constant γ depending only on n and M it follows that

$$\sum_{\mathbf{v} \in \mathbf{Z}^n \setminus \{\mathbf{0}\}} |\partial^{\boldsymbol{\alpha}} \mathscr{F}\eta(\sqrt{\mathscr{D}}\, \mathbf{v})|$$
$$\leqslant \sum_{\mathbf{v} \in \mathbf{Z}^n \setminus \{\mathbf{0}\}} |\partial^{\boldsymbol{\alpha}} \mathscr{F}\eta(\sqrt{\mathscr{D}}\, \mathbf{v})| \left(\sum_{[\boldsymbol{\beta}]=M} (2\pi \mathbf{v})^{2\boldsymbol{\beta}} \right)^{1/2} \gamma^{-1/2}\, |\mathbf{v}|^{-M}$$
$$\leqslant \gamma^{-1/2} \left(\sum_{\mathbf{v} \in \mathbf{Z}^n \setminus \{\mathbf{0}\}} |\partial^{\boldsymbol{\alpha}} \mathscr{F}\eta(\sqrt{\mathscr{D}}\, \mathbf{v})|^2 \sum_{[\boldsymbol{\beta}]=M} (2\pi \mathbf{v})^{2\boldsymbol{\beta}} \right)^{1/2}$$
$$\times \left(\sum_{\mathbf{v} \in \mathbf{Z}^n \setminus \{\mathbf{0}\}} |\mathbf{v}|^{-2M} \right)^{1/2}$$
$$\leqslant \gamma^{-1/2} c^{1/2} \mathscr{D}^{-M/2} \left(\sum_{\mathbf{v} \in \mathbf{Z}^n \setminus \{\mathbf{0}\}} |\mathbf{v}|^{-2M} \right)^{1/2}. \qquad \blacksquare$$

THEOREM 2.1. *Suppose that the generating function η satisfies the assumptions of Lemma* 2.4 *and the moment condition* (1.2). *If the function u is as in Lemma* 2.2 *then for any $\varepsilon > 0$ there exists $\mathscr{D} > 0$ such that*

$$|\mathscr{M}_h u(\mathbf{x}) - u(\mathbf{x})| \leqslant c(\sqrt{\mathscr{D}}\, h)^N \sum_{[\boldsymbol{\alpha}]=N} \frac{\|\partial^{\boldsymbol{\alpha}} u\|_{C(B(\mathbf{x},\kappa))}}{\boldsymbol{\alpha}!} + \varepsilon \sum_{[\boldsymbol{\alpha}]=0}^{N-1} (\sqrt{\mathscr{D}}\, h)^{[\boldsymbol{\alpha}]} \frac{|\partial^{\boldsymbol{\alpha}} u(\mathbf{x})|}{\boldsymbol{\alpha}!}, \tag{2.5}$$

where the constant c depends only on η and κ.

Thus the quasi-interpolant (2.2) approximates a smooth function u with the order $\mathcal{O}((\sqrt{\mathscr{D}}\, h)^N)$ up to some saturation error. By choosing the parameter $\mathscr{D}$ large enough, this error can be made negligible in numerical computations.

In fact any sufficiently smooth and rapidly decaying function η nonorthogonal to 1 can be used as basis for quasi-interpolation formulas providing estimates of the type (2.5) with arbitrarily large N. To ensure that $\mathscr{F}\eta - 1$ has a zero of order N in $\mathbf{0}$ one can apply well-known techniques to improve the approximation order of quasi-interpolants (see e.g. [14]). We give here another analytic formula. For notational convenience we denote $\partial^{\boldsymbol{\alpha}}(\mathscr{F}\eta)^{-1}(\mathbf{0}) := \partial^{\boldsymbol{\alpha}}(1/\mathscr{F}\eta(\boldsymbol{\lambda}))|_{\boldsymbol{\lambda}=\mathbf{0}}$.

LEMMA 2.5. *Suppose that η satisfies* (2.1), *is $(N-1)$-times continuously differentiable and that*

$$\int_{\mathbf{R}^n} \eta(\mathbf{t})\, d\mathbf{t} \neq 0, \qquad \int_{\mathbf{R}^n} |\mathbf{t}|^{N-1} |\partial^{\boldsymbol{\alpha}} \eta(\mathbf{t})|\, d\mathbf{t} < \infty, \qquad 0 \leqslant |\boldsymbol{\alpha}| \leqslant N-1.$$

Then

$$\eta_N(\mathbf{x}) = \sum_{[\boldsymbol{\alpha}]=0}^{N-1} \frac{\partial^{\boldsymbol{\alpha}}(\mathscr{F}\eta)^{-1}(\mathbf{0})}{\boldsymbol{\alpha}!\,(2\pi i)^{[\boldsymbol{\alpha}]}}\, \partial^{\boldsymbol{\alpha}} \eta(\mathbf{x}) \tag{2.6}$$

satisfies the moment condition (1.2).

Proof. Denote by P_N the Nth order Taylor polynomial of $1/\mathscr{F}\eta$

$$P_N(\boldsymbol{\lambda}) = \sum_{[\boldsymbol{\alpha}]=0}^{N-1} \partial^{\boldsymbol{\alpha}}(\mathscr{F}\eta)^{-1}(\mathbf{0}) \frac{\boldsymbol{\lambda}^{\boldsymbol{\alpha}}}{\boldsymbol{\alpha}!}.$$

Then

$$\partial^{\boldsymbol{\beta}}(P_N(\boldsymbol{\lambda})\, \mathscr{F}\eta(\boldsymbol{\lambda}))|_{\boldsymbol{\lambda}=\mathbf{0}} = \partial^{\boldsymbol{\beta}} \left(\frac{1}{\mathscr{F}\eta(\boldsymbol{\lambda})} \mathscr{F}\eta(\boldsymbol{\lambda}) \right)\bigg|_{\boldsymbol{\lambda}=\mathbf{0}} = \delta_{[\boldsymbol{\beta}]0}$$

for all $[\boldsymbol{\beta}] \leqslant N-1$, and

$$P_N(\boldsymbol{\lambda})\, \mathscr{F}\eta(\boldsymbol{\lambda}) = \mathscr{F}\left(\sum_{[\boldsymbol{\alpha}]=0}^{N-1} \partial^{\boldsymbol{\alpha}}(\mathscr{F}\eta)^{-1}(\mathbf{0})\, \frac{1}{\boldsymbol{\alpha}!} \left(\frac{1}{2\pi i}\right)^{[\boldsymbol{\alpha}]} \partial^{\boldsymbol{\alpha}}\eta(\mathbf{x}) \right)(\boldsymbol{\lambda}). \quad \blacksquare$$

In the special case of a radial generating function, i.e. $\eta(\mathbf{x}) = \psi(|\mathbf{x}|^2)$, the moment condition is equivalent to

$$\mathscr{F}\eta(\mathbf{0}) = 1, \quad \Delta^k \mathscr{F}\eta(\mathbf{0}) = 0, \qquad k = 1, ..., M-1,$$

with the Laplace operator Δ. If one chooses the polynomial

$$P_{2M}(\boldsymbol{\lambda}) = \sum_{j=0}^{M-1} \frac{\Gamma\left(\frac{n}{2}\right) \Delta^j (\mathscr{F}\eta)^{-1}(\mathbf{0})}{j!\, 2^{2j}\, \Gamma\left(j+\frac{n}{2}\right)} |\boldsymbol{\lambda}|^{2j},$$

which satisfies

$$\Delta^k P_{2M}(\mathbf{0}) = \Delta^k (\mathscr{F}\eta)^{-1}(\mathbf{0}) := \Delta^k \left.\frac{1}{\mathscr{F}\eta(\boldsymbol{\xi})}\right|_{\boldsymbol{\xi}=\mathbf{0}}, \qquad k = 0, ..., M-1,$$

then it follows

LEMMA 2.6. *Suppose that η satisfies the conditions of Lemma 2.5 and is additionally a radial function. Then the function*

$$\Gamma\left(\frac{n}{2}\right) \sum_{j=0}^{M-1} \frac{(-1)^j\, \Delta^j (\mathscr{F}\eta)^{-1}(\mathbf{0})}{j!\, (4\pi)^{2j}\, \Gamma\left(j+\frac{n}{2}\right)} \Delta^j \eta(\mathbf{x}) \tag{2.7}$$

satisfies the moment conditions (1.2) *with $N = 2M$.*

An interesting feature of formula (2.7) is its additive structure. The order of a given quasi-interpolant can be increased by adding a new formula (2.2) with the next term of (2.7) as generating function.

To obtain a concrete function system we apply (2.7) to the Gaussian $\eta(\mathbf{x}) = \exp(-|\mathbf{x}|^2)$. From

$$\Delta^j e^{-|\mathbf{x}|^2} = \left(\frac{1}{r^{n-1}} \frac{d}{dr} r^{n-1} \frac{d}{dr}\right)^j e^{-r^2} = 4^j \left(y \frac{d^2}{dy^2} + \frac{n}{2} \frac{d}{dy}\right)^j e^{-y} \Big|_{y=|\mathbf{x}|^2}$$

and

$$\Delta^j e^{\pi^2 |\xi|^2}\big|_{\xi=\mathbf{0}} = \frac{(4\pi^2)^j \, \Gamma\left(j+\frac{n}{2}\right)}{\Gamma\left(\frac{n}{2}\right)}$$

we see that

$$\eta_{2M}(\mathbf{x}) = \Gamma\left(\frac{n}{2}\right) \sum_{j=0}^{M-1} \frac{(-1)^j}{j!\,(4\pi)^{2j}\,\Gamma\left(j+\frac{n}{2}\right)} \,\Delta^j e^{\pi^2|\xi|^2}\Big|_{\xi=\mathbf{0}} \Delta^j e^{-|\mathbf{x}|^2}$$

$$= \sum_{j=0}^{M-1} \frac{(-1)^j}{j!} \left(y\,\frac{d^2}{dy^2} + \frac{n}{2}\,\frac{d}{dy}\right)^j e^{-y}\Big|_{y=|\mathbf{x}|^2} = p_{M-1}(|\mathbf{x}|^2)\, e^{-|\mathbf{x}|^2}$$

with a polynomial p_{M-1} of degree $M-1$. Since

$$\int_{\mathbf{R}^n} |\mathbf{x}|^{2k}\, \eta_{2M}(\mathbf{x})\, d\mathbf{x} = \frac{\pi^{n/2}}{\Gamma(n/2)} \int_0^\infty y^{k-1} p_{M-1}(y)\, y^{n/2} e^{-y}\, dy = \delta_{k0}$$

for $0 \leqslant k \leqslant M-1$, the polynomial p_{M-1} is orthogonal to $\{y^k\}_{k=0}^{M-2}$ with respect to the weight $y^{n/2}e^{-y}$. Hence $\{p_{M-1}\}$ are the generalized Laguerre polynomials ([1]), more precisely

$$p_{M-1}(y) = \pi^{-n/2} L_{M-1}^{(n/2)}(y) \qquad \text{with} \quad L_M^{(\gamma)}(y) = \frac{e^y y^{-\gamma}}{M!} \left(\frac{d}{dy}\right)^M (e^{-y} y^{M+\gamma}).$$

THEOREM 2.2. *An n-dimensional quasi-interpolation formula* (2.2) *providing the error estimate* (2.5) *with* $N=2M$ *is given by the generating function*

$$\eta_{2M}(\mathbf{x}) = \pi^{-n/2} L_{M-1}^{(n/2)}(|\mathbf{x}|^2)\, e^{-|\mathbf{x}|^2} \tag{2.8}$$

having the Fourier transform

$$\mathscr{F}\eta_{2M}(\boldsymbol{\lambda}) = e^{-\pi^2|\boldsymbol{\lambda}|^2} \sum_{j=0}^{M-1} \frac{(\pi^2|\boldsymbol{\lambda}|^2)^j}{j!}.$$

Note that for $\mathscr{D}=4$ the leading term of the saturation error

$$\sum_{\mathbf{v}\in\mathbf{Z}^n\setminus\{\mathbf{0}\}} |\mathscr{F}\eta_{2M}(\sqrt{\mathscr{D}}\,\mathbf{v})|$$

is comparable to the machine epsilon for REAL*8-arithmetics. Because the classical potentials of mathematical physics with the density η_{2M} are known analytic expressions they lead to high order semi-analytic cubature formulas (see [8, 10]).

Theorem 2.1 indicates that it is possible to obtain quasi-interpolants converging for $h\to 0$ if the parameter $\mathscr{D}$ is chosen depending on h. For example, taking $\eta=\eta_{2M}$ and $\mathscr{D}(h)=2M\,|\log h|/\pi^2$ it follows easily that (2.2) converges to u with the rate $\mathcal{O}((h\log h)^{2M})$. For special generating functions similar estimates were studied in [2, 3, 12] and possibly others. However, the nonstationary refinement enlarges for smaller h the number of summands necessary to compute the quasi-interpolant at a fixed point $\mathbf{x}$ within a given tolerance. Moreover, the use of stationary refinement in numerical computations has several advantages. For example, the cubature of a convolution operator $\mathscr{K}$ requires the evaluation of $\mathscr{K}\eta((\mathbf{x}/h-\mathbf{m})/\sqrt{\mathscr{D}})$. Hence, at the points $h\mathbf{k}$, $\mathbf{k}\in\mathbf{Z}^n$, these values are independent of h. Therefore, if the cubature is a part of some iterative or multiscale algorithm using different h one can precompute the values of $\mathscr{K}\eta(\mathbf{k}/\sqrt{\mathscr{D}})$ and use them at each level.

For the case of quite arbitrary generating functions the construction of highly accurate quasi-interpolants and their convergence properties are investigated in the recent paper [14]. It is shown, that starting with basis functions satisfying certain smoothness and decay conditions one can construct generating functions such that the order of the quasi-interpolation formula is limited by the smoothness of the basis function. This quasi-interpolant is a linear combination of h^2-translates of the given basis function.

3. NON-UNIFORMLY DISTRIBUTED CENTERS

The quasi-interpolation considered in the previous section can be generalized in a straightforward way to cases where the data points lie on a affine lattice of the form $hA\mathbf{Z}^n$ with a nonsingular $n\times n$-matrix A. In the following we study the more interesting case of quasi-interpolation formulas connected with nonuniform spacing of the grid points in some domain $\Omega\subset\mathbf{R}^n$ and also for functions given on an n-dimensional manifold. The approximant should have a simple semi-discrete convolutional form similar to (2.2) in order to get effective methods for computing pseudodifferential operators given on Ω.

We consider the following situation which occurs after applying a partition of unity. Suppose that $\omega \subset \mathbf{R}^n$ is a bounded domain and that there is defined a sufficiently smooth and non-singular mapping $\boldsymbol{\phi} = (\varphi_1, ..., \varphi_s)$: $\mathbf{R}^n \to \mathbf{R}^s$, $n \leqslant s$. That means, that

$$|\boldsymbol{\phi}'(\mathbf{y})| = \left(\sum_{(i)} (\kappa_{(i)}(\mathbf{y}))^2 \right)^{1/2} \neq 0, \qquad \mathbf{y} \in \omega, \tag{3.1}$$

where $\kappa^{(i)}$ denotes the minor of order n of the matrix

$$\boldsymbol{\phi}'(\mathbf{y}) = \left\| \begin{matrix} \partial\varphi_1/\partial y_1 & \cdots & \partial\varphi_1/\partial y_n \\ \vdots & \ddots & \vdots \\ \partial\varphi_s/\partial y_1 & \cdots & \partial\varphi_s/\partial y_n \end{matrix} \right\|,$$

corresponding to the rows with indices $i_1 < \cdots < i_n$. The sum is extended over all distinct $(i) = (i_1, ..., i_n)$, $1 \leqslant i_p \leqslant s$, of this kind. Then $\boldsymbol{\phi}$ generates a one-to-one mapping between ω and $\Omega = \boldsymbol{\phi}(\omega) \subset \mathbf{R}^s$.

Let u be an N-times continuously differentiable function on Ω with compact support, i.e. $u \circ \boldsymbol{\phi} \in C_0^N(\omega)$. In the following we study the approximation of u by the quasi-interpolant

$$u_h(\mathbf{x}) = \mathscr{D}^{-n/2} \sum_{h\mathbf{m} \in B} u(\boldsymbol{\phi}(h\mathbf{m}))\, \eta \left(\frac{\mathbf{x} - \boldsymbol{\phi}(h\mathbf{m})}{\sqrt{\mathscr{D}}\, h\, |\boldsymbol{\phi}'(y\mathbf{m})|^{1/n}} \right),$$

$$\mathbf{x} = \boldsymbol{\phi}(\mathbf{y}) \in \Omega, \tag{3.2}$$

where the summation extends over all $h\mathbf{m}$ belonging to some subset $B \subset \omega$ depending on the given point $\mathbf{x} \in \Omega$.

Note that in the case $s = n$ formula (3.2) corresponds to quasi-interpolation on a domain $\Omega \subset \mathbf{R}^n$ with respect to a set of data points $\{\mathbf{x}_\mathbf{m}\}$ which can be represented as sufficiently smooth image of a uniform lattice, $\mathbf{x}_\mathbf{m} = \boldsymbol{\phi}(h\mathbf{m})$, $h\mathbf{m} \in \omega$. If $s > n$ then $\Omega = \boldsymbol{\phi}(\omega) \subset \mathbf{R}^s$ can be considered as part of a n-dimensional manifold parametrized by $\boldsymbol{\phi}$. The generating function η is given in $\mathbf{R}^s$, hence (3.2) defines a function in $\mathbf{R}^s$. We are interested in how the restriction of this linear combination to Ω approximates the function u on Ω. In the following we suppose that in the case $s > n$ the generating function η is radial.

We will show that the difference $u_h - u$ has a similar behavior as in the case considered in Section 2 provided η is N-times continuously differentiable, satisfies the moment condition (1.2) and the decay condition

$$|\partial^{\boldsymbol{\alpha}} \eta(\mathbf{y})| \leqslant A_{K, \boldsymbol{\alpha}} (1 + |\mathbf{y}|^2)^{-([\boldsymbol{\alpha}] + K)/2}, \qquad \forall \boldsymbol{\alpha}, \quad 0 \leqslant [\boldsymbol{\alpha}] \leqslant N, \quad \mathbf{y} \in \mathbf{R}^n, \tag{3.3}$$

for some number $K > N + n$ and constants $A_{K,\boldsymbol{\alpha}}$. Since for $s > n$ the generating function η is supposed to be radial the conditions (1.2) and (3.3) posed in the space $\mathbf{R}^n$ make sense.

Remark. For a given non-singular parametrization $\boldsymbol{\phi}$ of Ω the quasi-interpolant

$$\mathscr{D}^{-n/2} \sum_{m \in \mathbf{Z}^n} u(\boldsymbol{\phi}(h\mathbf{m}))\, \eta\left(\frac{\boldsymbol{\phi}^{-1}(\mathbf{x}) - h\mathbf{m}}{\sqrt{\mathscr{D}}\, h}\right)$$

provides of course the estimate (2.5). However, we want to have approximations with the property that the action of convolution operators, for example, can be effectively determined. These integral operators have difference kernels with respect to the variable $\mathbf{x} \in \Omega$, therefore formula (3.2) leads to semi-analytic approximations as soon as the convolution of η is known.

The approximation property of (3.2) is formulated in the following theorem.

THEOREM 3.1. *Assume besides the decay and moment conditions* (3.3) *and* (1.2) *that η is continuously differentiable of order $N + M - 1$ in $\mathbf{R}^n$ with M the smallest integer greater $n/2$. Additionally assume, that all derivatives $\partial^{\boldsymbol{\beta}}\eta$, $N < [\boldsymbol{\beta}] \leqslant N + M - 1$, satisfy the condition*

$$|\partial^{\boldsymbol{\beta}}\eta(\mathbf{y})| \leqslant A_{K,\boldsymbol{\beta}}(1 + |\mathbf{y}|^2)^{-(N+K)/2}, \tag{3.4}$$

and that $\boldsymbol{\phi}: \omega \to \Omega$ is in the class C^{N+1}. If $u \in C_0^N(\Omega)$, then for any $\varepsilon > 0$ there exists $\mathscr{D} > 0$ such that at any point $\mathbf{x} \in \Omega$

$$|u_h(\mathbf{x}) - u(\mathbf{x})| \leqslant c(\sqrt{\mathscr{D}}\, h)^N \, \|u\|_{C^N(\bar{\Omega})} + \varepsilon \sum_{k=0}^{N-1} c_k(\sqrt{\mathscr{D}}\, h)^k, \tag{3.5}$$

where c does not depend on u, h and $\mathscr{D}$ and the numbers c_k can be obtained from the values $\partial^{\boldsymbol{\alpha}}u(\mathbf{x})$, $[\boldsymbol{\alpha}] \leqslant k$.

The proof is based on some lemmas. Let us fix a point $\mathbf{x} \in \Omega$, denote $\mathbf{y}_0 = \boldsymbol{\phi}^{-1}(\mathbf{x}) \in \omega$ and make the substitution

$$\boldsymbol{\xi}(\mathbf{y}) = \frac{\mathbf{x} - \boldsymbol{\phi}(\mathbf{y})}{|\boldsymbol{\phi}'(\mathbf{y})|^{1/n}} = \frac{\boldsymbol{\phi}(\mathbf{y}_0) - \boldsymbol{\phi}(\mathbf{y})}{|\boldsymbol{\phi}'(\mathbf{y})|^{1/n}}, \qquad \mathbf{y} \in \omega. \tag{3.6}$$

LEMMA 3.1. *The mapping $\boldsymbol{\xi}: \omega \to \mathbf{R}^s$ can be represented in the form*

$$\boldsymbol{\xi}(\mathbf{y}) = A(\mathbf{y}_0 - \mathbf{y}) + |\mathbf{y}_0 - \mathbf{y}|^2\, \tilde{\boldsymbol{\xi}}(\mathbf{y}), \qquad \mathbf{y} \in \omega, \tag{3.7}$$

where $A\colon \mathbf{R}^n \to \mathbf{R}^s$ *is a linear mapping with* $|A| = 1$ *and* $\tilde{\xi} \in C^{N-1}$. *There exist a closed ball* $B(\mathbf{y}_0, \kappa) \subset \omega$ *centered at* $\mathbf{y}_0$ *with radius* $\kappa > 0$ *and positive constants* C_1 *and* C_2 *such that for any* $\mathbf{y} \in B(\mathbf{y}_0, \kappa)$ *and all real* $s \in [0, 1]$

$$|\xi'(\mathbf{y})| \geqslant C_1 \qquad \text{and} \qquad |sA(\mathbf{y}_0 - \mathbf{y}) + (1-s)\, \xi(\mathbf{y})| \geqslant C_2\, |\mathbf{y}_0 - \mathbf{y}|. \tag{3.8}$$

Proof. Since

$$\xi'(\mathbf{y}) = -\frac{\boldsymbol{\phi}'(\mathbf{y})}{|\boldsymbol{\phi}'(\mathbf{y})|^{1/n}} - \frac{\xi(\mathbf{y})(\nabla\, |\boldsymbol{\phi}'(\mathbf{y})|)^T}{n\, |\boldsymbol{\phi}'(\mathbf{y})|}$$

and $\xi(\mathbf{y}_0) = \mathbf{0}$ we obtain $A = |\boldsymbol{\phi}'(\mathbf{y}_0)|^{-1/n}\, \boldsymbol{\phi}'(\mathbf{y}_0)$ such that $|A| = 1$. Hence the matrix A^*A is not singular and therefore $|A\mathbf{y}| \geqslant c\, |\mathbf{y}|$ with some positive constant c. Now Taylor's formula leads to (3.7) and to

$$\begin{aligned} |sA(\mathbf{y}_0 - \mathbf{y}) + (1-s)\, \xi(\mathbf{y})| &= |A(\mathbf{y}_0 - \mathbf{y}) + (1-s)\, |\mathbf{y}_0 - \mathbf{y}|^2\, \tilde{\xi}(\mathbf{y})| \\ &\geqslant (c - |\mathbf{y}_0 - \mathbf{y}|\, |\tilde{\xi}(\mathbf{y})|)\, |\mathbf{y}_0 - \mathbf{y}|. \quad \blacksquare \end{aligned}$$

After having fixed the ball $B(\mathbf{y}_0, \kappa)$ we study in the following the quasi-interpolant

$$u_h(\mathbf{x}) = \mathscr{D}^{-n/2} \sum_{h\mathbf{m} \in B(\mathbf{y}_0, \kappa)} u(\boldsymbol{\phi}(h\mathbf{m}))\, \eta\left(\frac{\xi(h\mathbf{m})}{\sqrt{\mathscr{D}}\, h}\right). \tag{3.9}$$

We will give an asymptotic expansion of $u_h(\mathbf{x})$ for $h \to 0$ up to terms of the order $\mathcal{O}(h^N)$. By using (3.7) and the Taylor expansion of η we split $u_h(\mathbf{x})$ into a finite sum of semi-discrete convolutions plus a remaining term. In the following we denote the variables in $\mathbf{R}^s$ by $\mathbf{x}$, whereas the symbol $\mathbf{y}$ denotes variables in $\omega \in \mathbf{R}^n$. Thus the Taylor expansion of the function η on $\mathbf{R}^s$ around the point $A(\mathbf{y}_0 - \mathbf{y})/\sqrt{\mathscr{D}}\, h$ yields

$$\begin{aligned} \eta\left(\frac{\xi(\mathbf{y})}{\sqrt{\mathscr{D}}\, h}\right) &= \sum_{[\boldsymbol{\beta}]=0}^{N-1} \frac{(|\mathbf{y}_0 - \mathbf{y}|^2\, \tilde{\xi}(\mathbf{y}))^{\boldsymbol{\beta}}}{\boldsymbol{\beta}!\, (\sqrt{\mathscr{D}}\, h)^{[\boldsymbol{\beta}]}}\, \partial_{\mathbf{x}}^{\boldsymbol{\beta}} \eta\left(\frac{A(\mathbf{y}_0 - \mathbf{y})}{\sqrt{\mathscr{D}}\, h}\right) \\ &\quad + \frac{N}{(\sqrt{\mathscr{D}}\, h)^N} \sum_{[\boldsymbol{\beta}]=N} \frac{(|\mathbf{y}_0 - \mathbf{y}|^2\, \tilde{\xi}(\mathbf{y}))^{\boldsymbol{\beta}}}{\boldsymbol{\beta}!} \\ &\quad \times \int_0^1 s^{N-1}\, \partial_{\mathbf{x}}^{\boldsymbol{\beta}} \eta\left(\frac{sA(\mathbf{y}_0 - \mathbf{y}) + (1-s)\, \xi(\mathbf{y})}{\sqrt{\mathscr{D}}\, h}\right) ds, \end{aligned}$$

where $\partial_{\mathbf{x}}^{\boldsymbol{\beta}}$ denotes the corresponding partial derivatives in $\mathbf{R}^s$. Using the notations

$$\tilde{u}(\mathbf{y}) = u(\boldsymbol{\phi}(\mathbf{y})) \qquad \text{and} \qquad \tilde{\eta}_{A,\boldsymbol{\beta}}(\mathbf{y}) = |\mathbf{y}|^{2[\boldsymbol{\beta}]}\, \partial_{\mathbf{x}}^{\boldsymbol{\beta}} \eta(A\mathbf{y})$$

we obtain the splitting of the quasi-interpolant (3.9)

$$\mathscr{D}^{-n/2} \sum_{h\mathbf{m} \in B(\mathbf{y}_0, \kappa)} u(\boldsymbol{\phi}(h\mathbf{m}))\, \eta\left(\frac{\boldsymbol{\xi}(h\mathbf{m})}{\sqrt{\mathscr{D}}\, h}\right)$$

$$= \sum_{[\boldsymbol{\beta}]=0}^{N-1} \frac{(\sqrt{\mathscr{D}}\, h)^{[\boldsymbol{\beta}]}}{\boldsymbol{\beta}!\, \mathscr{D}^{n/2}} \sum_{h\mathbf{m} \in B(\mathbf{y}_0, \kappa)} \tilde{u}(h\mathbf{m})\, \tilde{\boldsymbol{\xi}}(h\mathbf{m})^{\boldsymbol{\beta}}\, \tilde{\eta}_{A,\boldsymbol{\beta}}\left(\frac{\mathbf{y}_0 - h\mathbf{m}}{\sqrt{\mathscr{D}}\, h}\right) + R_N(\mathbf{y}_0), \tag{3.10}$$

where the remaining term is of the form

$$R_N(\mathbf{y}_0) = N(\sqrt{\mathscr{D}}\, h)^N \sum_{[\boldsymbol{\beta}]=N} \frac{1}{\boldsymbol{\beta}!} \int_0^1 s^{N-1}$$

$$\times \mathscr{D}^{-n/2} \sum_{h\mathbf{m} \in B(\mathbf{y}_0, \kappa)} \tilde{u}(h\mathbf{m})\, \tilde{\boldsymbol{\xi}}(h\mathbf{m})^{\boldsymbol{\beta}}$$

$$\times \frac{|\mathbf{y}_0 - h\mathbf{m}|^{2N}}{(\mathscr{D}h^2)^N}\, \partial_{\mathbf{x}}^{\boldsymbol{\beta}} \eta\left(\frac{sA(\mathbf{y}_0 - h\mathbf{m}) + (1-s)\, \boldsymbol{\xi}(h\mathbf{m})}{\sqrt{\mathscr{D}}\, h}\right) ds.$$

LEMMA 3.2. *Suppose that η satisfies the decay condition* (3.3). *Then*

$$|R_N(\mathbf{y}_0)| \leqslant c(\sqrt{\mathscr{D}}\, h)^N\, \|\tilde{u}\|_{C(B(\mathbf{y}_0, \kappa))}$$

with a constant c depending only on η and $\boldsymbol{\phi}$.

Proof. In view of (3.3), (3.8) and $[\boldsymbol{\beta}] = N$

$$\Bigg| \mathscr{D}^{-n/2} \sum_{h\mathbf{m} \in B(\mathbf{y}_0, \kappa)} \tilde{u}(h\mathbf{m})\, \tilde{\boldsymbol{\xi}}(h\mathbf{m})^{\boldsymbol{\beta}} \frac{|\mathbf{y}_0 - h\mathbf{m}|^{2N}}{(\mathscr{D}h^2)^N}$$

$$\times \partial_{\mathbf{x}}^{\boldsymbol{\beta}} \eta\left(\frac{sA(\mathbf{y}_0 - h\mathbf{m}) + (1-s)\, \boldsymbol{\xi}(h\mathbf{m})}{\sqrt{\mathscr{D}}\, h}\right)\Bigg|$$

$$\leqslant cA_{K,\boldsymbol{\beta}}\mathscr{D}^{-n/2} \sum_{h\mathbf{m} \in B(\mathbf{y}_0, \kappa)} \frac{|\mathbf{y}_0 - h\mathbf{m}|^{2N}}{(\mathscr{D}h^2)^N}$$

$$\times \left(1 + \frac{|sA(\mathbf{y}_0 - h\mathbf{m}) + (1-s)\, \boldsymbol{\xi}(h\mathbf{m})|^2}{\mathscr{D}h^2}\right)^{-(N+K)/2}$$

$$\leqslant c A_{K,\boldsymbol{\beta}} \mathscr{D}^{-n/2} \sum_{h\mathbf{m} \in B(\mathbf{y}_0, \kappa)} \left(\frac{|\mathbf{y}_0 - h\mathbf{m}|^2}{\mathscr{D} h^2} \right)^N \times \left(1 + \frac{(C_2 |\mathbf{y}_0 - h\mathbf{m}|)^2}{\mathscr{D} h^2} \right)^{-(N+K)/2}$$

$$\leqslant c A_{K,\boldsymbol{\beta}} C_2^{-2N} \mathscr{D}^{-n/2} \sum_{h\mathbf{m} \in B(\mathbf{y}_0, \kappa)} \left(1 + \frac{(C_2 |\mathbf{y}_0 - h\mathbf{m}|)^2}{\mathscr{D} h^2} \right)^{(N-K)/2}.$$

The last sum is the simple midpoint cubature of the integral

$$(\sqrt{\mathscr{D}}\, h)^{-n} \int_{B(\mathbf{y}_0, \kappa)} \left(1 + \frac{(C_2 |\mathbf{y}_0 - \mathbf{y}|)^2}{\mathscr{D} h^2} \right)^{(N-K)/2} d\mathbf{y}$$

$$= \int_{B(\kappa/\sqrt{\mathscr{D}}\, h)} (1 + (C_2 |\mathbf{t}|)^2)^{(N-K)/2}\, d\mathbf{t},$$

which is because of $K > N + n$ uniformly bounded for all positive $\mathscr{D}$ and h. It is well known that for any $g \in C^1(B(\mathbf{y}_0, \kappa))$ the cubature error can be estimated by

$$\left| h^n \sum_{h\mathbf{m} \in B(\mathbf{y}_0, \kappa)} g(h\mathbf{m}) - \int_{U(\mathbf{y}_0)} g(\mathbf{y})\, d\mathbf{y} \right| \leqslant c h \, \|g\|_{C^1(B(\mathbf{y}_0, \kappa))}.$$

Since the function

$$g(\mathbf{y}) = (\sqrt{\mathscr{D}}\, h)^{-n} \left(1 + \frac{(C_2 |\mathbf{y}_0 - \mathbf{y}|)^2}{\mathscr{D} h^2} \right)^{(N-K)/2}$$

satisfies $\|g\|_{C^1(B(\mathbf{y}_0, \kappa))} \leqslant c(\sqrt{\mathscr{D}}\, h)^{-1}$ the assertion follows. ∎

Now we consider the semi-discrete convolutions in (3.9). First note that in view of (3.3) the generating functions $\eta_{A,\boldsymbol{\beta}}$ satisfy the decay condition (2.1) with $K > N - [\boldsymbol{\beta}] + n$. Hence we can apply Lemma 2.2 to the functions $\tilde{u} \boldsymbol{\xi}^{\boldsymbol{\beta}} \in C^{N - [\boldsymbol{\beta}]}$ and to the sum

$$\mathscr{D}^{-n/2} \sum_{h\mathbf{m} \in B(\mathbf{y}_0, \kappa)} \tilde{u}(h\mathbf{m})\, \boldsymbol{\xi}(h\mathbf{m})^{\boldsymbol{\beta}}\, \tilde{\eta}_{A,\boldsymbol{\beta}} \left(\frac{\mathbf{y}_0 - h\mathbf{m}}{\sqrt{\mathscr{D}}\, h} \right).$$

Since condition (3.4) guarantees that Lemmas 2.3 and 2.4 can be applied we obtain

LEMMA 3.3.

$$\frac{(\sqrt{\mathscr{D}}\,h)^{[\boldsymbol{\beta}]}}{\mathscr{D}^{n/2}} \sum_{h\mathbf{m} \in B(\mathbf{y}_0, \kappa)} \tilde{u}(h\mathbf{m})\, \tilde{\xi}(h\mathbf{m})^{\boldsymbol{\beta}}\, \tilde{\eta}_{A, \boldsymbol{\beta}}\left(\frac{\mathbf{y}_0 - h\mathbf{m}}{\sqrt{\mathscr{D}}\,h}\right)$$

$$= \sum_{[\boldsymbol{\alpha}]=0}^{N-[\boldsymbol{\beta}]-1} \frac{\partial_{\mathbf{y}}^{\boldsymbol{\alpha}}(\tilde{u}\tilde{\xi}^{\boldsymbol{\beta}})(\mathbf{y}_0)}{\boldsymbol{\alpha}!} \frac{(\sqrt{\mathscr{D}}\,h)^{[\boldsymbol{\alpha}]+[\boldsymbol{\beta}]}}{(2\pi i)^{[\boldsymbol{\alpha}]}}$$

$$\times \sum_{\mathbf{v} \in \mathbf{Z}^n} \partial^{\boldsymbol{\alpha}} \mathscr{F} \tilde{\eta}_{A, \boldsymbol{\beta}}(\sqrt{\mathscr{D}}\,\mathbf{v})\, e_{\mathbf{v}}(\mathbf{y}_0/h) + R_{N-\boldsymbol{\beta}}(\mathbf{y}_0),$$

with the remainder bounded by

$$|R_{N-\boldsymbol{\beta}}(\mathbf{y})| \leqslant c(\sqrt{\mathscr{D}}\,h)^N \,\|\nabla_{N-[\boldsymbol{\beta}]}(\tilde{u}\tilde{\xi}^{\boldsymbol{\beta}})\|_{C(B(\mathbf{y}_0, \kappa))}.$$

Thus the behavior of the quasi-interpolant (3.9) can be derived from the values of the partial derivatives of the n-dimensional Fourier transform

$$\partial^{\boldsymbol{\alpha}} \mathscr{F} \tilde{\eta}_{A, \boldsymbol{\beta}}, \qquad 0 \leqslant [\boldsymbol{\beta}] \leqslant N-1, \quad 0 \leqslant [\boldsymbol{\alpha}] \leqslant N-[\boldsymbol{\beta}]-1,$$

at the points $\{\sqrt{\mathscr{D}}\,\mathbf{v},\, \mathbf{v} \in \mathbf{Z}^n\}$.

LEMMA 3.4. *Let $s = n$. Then*

$$\partial^{\boldsymbol{\alpha}} \mathscr{F} \tilde{\eta}_{A, \boldsymbol{\beta}}(\boldsymbol{\lambda}) = (2\pi i)^{-[\boldsymbol{\beta}]}\, \Delta^{[\boldsymbol{\beta}]}\, \partial^{\boldsymbol{\alpha}}(\boldsymbol{\lambda}^{\boldsymbol{\beta}} \mathscr{F} \eta(A^{-1}\boldsymbol{\lambda})).$$

In particular,

$$\partial^{\boldsymbol{\alpha}} \mathscr{F} \tilde{\eta}_{A, \boldsymbol{\beta}}(\mathbf{0}) = \begin{cases} 1, & [\boldsymbol{\alpha}] = [\boldsymbol{\beta}] = 0 \\ 0, & 1 \leqslant [\boldsymbol{\beta}] + [\boldsymbol{\alpha}] \leqslant N-1. \end{cases}$$

Collecting the results of Lemmas 3.2, 3.3, 3.4 together with (3.10) we obtain that the quasi-interpolant (3.9) defined in the domain $\Omega \subset \mathbf{R}^n$ has the following representation

$$u_h(\mathbf{x}) = u(\mathbf{x}) + \sum_{[\boldsymbol{\beta}]=0}^{N-1} \sum_{[\boldsymbol{\alpha}]=0}^{N-[\boldsymbol{\beta}]-1} \frac{(\sqrt{\mathscr{D}}\,h)^{[\boldsymbol{\alpha}]+[\boldsymbol{\beta}]}}{\boldsymbol{\alpha}!\, \boldsymbol{\beta}!\, (2\pi i)^{[\boldsymbol{\alpha}]+\boldsymbol{\beta}}} \partial_{\mathbf{y}}^{\boldsymbol{\alpha}}(\tilde{u}\tilde{\xi}^{\boldsymbol{\beta}})(\mathbf{y}_0)$$

$$\times \sum_{\mathbf{v} \in \mathbf{Z}^n \setminus \{\mathbf{0}\}} e_{\mathbf{v}}(\mathbf{y}_0/h)\, \Delta^{[\boldsymbol{\beta}]} \partial^{\boldsymbol{\alpha}}(\boldsymbol{\lambda}^{\boldsymbol{\beta}} \mathscr{F} \eta(A^{-1}\boldsymbol{\lambda}))|_{\boldsymbol{\lambda}=\sqrt{\mathscr{D}}\,\mathbf{v}} + O((\mathscr{D}h)^N),$$

which proves Theorem 3.1 in the case $s = n$.

Now we consider the case $s > n$, where the generating function η is supposed to depend smoothly on the norm $|\mathbf{x}|$, $\mathbf{x} \in \mathbf{R}^s$.

LEMMA 3.5. *Let* $A\colon \mathbf{R}^n \to \mathbf{R}^s$ *be a linear mapping of rank* n *and* η *is a smooth radial function in* $\mathbf{R}^s$. *Then for any multiindices* $\boldsymbol{\alpha} \in \mathbf{Z}^n_{\geqslant 0}$ *and* $\boldsymbol{\beta} \in \mathbf{Z}^s_{\geqslant 0}$ *with* $[\boldsymbol{\alpha}] = [\boldsymbol{\beta}]$ *there exist a collection of multiindices* $\boldsymbol{\gamma}$, $\boldsymbol{\delta} \in \mathbf{Z}^n_{\geqslant 0}$ *and numbers* $c_{\boldsymbol{\gamma},\boldsymbol{\delta}}$ *such that for all* $\mathbf{y} \in \mathbf{R}^n$

$$\mathbf{y}^{\boldsymbol{\alpha}}\, \partial^{\boldsymbol{\beta}}_{\mathbf{x}} \eta(A\mathbf{y}) = \sum_{[\boldsymbol{\delta}]=1}^{[\boldsymbol{\beta}]} \sum_{[\boldsymbol{\gamma}]=[\boldsymbol{\delta}]} c_{\boldsymbol{\gamma},\boldsymbol{\delta}}\, \mathbf{y}^{\boldsymbol{\gamma}}\, \partial^{\boldsymbol{\delta}}_{\mathbf{y}} \eta(A\mathbf{y}). \tag{3.11}$$

Proof. Since $A\mathbf{y} = \mathbf{x}$ with an $s \times n$ matrix of rank n there exist an invertible $n \times n$ matrix B and a subvector $\mathbf{x}'$ of length n such that $\mathbf{y} = B\mathbf{x}'$. For definiteness suppose the ordering $\mathbf{x} = (\mathbf{x}', \mathbf{x}'')$ and $\partial^{\boldsymbol{\beta}}_{\mathbf{x}} = \partial^{\boldsymbol{\beta}'}_{\mathbf{x}'} \partial^{\boldsymbol{\beta}''}_{\mathbf{x}''}$. For any $[\boldsymbol{\beta}'] = 1$ the partial derivative $\partial^{\boldsymbol{\beta}'}_{\mathbf{x}'}$ is a linear combination of $\partial^{\boldsymbol{\delta}}_{\mathbf{y}}$ with $[\boldsymbol{\delta}] = 1$. Hence, if the multiindex $\boldsymbol{\beta}'' \in \mathbf{Z}^{s-n}_{\geqslant 0}$ satisfies $\boldsymbol{\beta}'' = \mathbf{0}$ then (3.11) is true with $c_{\boldsymbol{\gamma},\boldsymbol{\delta}} = 0$ for all $[\boldsymbol{\delta}] < [\boldsymbol{\beta}]$.

Let $[\boldsymbol{\beta}''] \geqslant 1$. Since $\mathbf{y} = B\mathbf{x}'$ implies

$$\mathbf{y}^{\boldsymbol{\alpha}} = \sum_{[\boldsymbol{\delta}]=[\boldsymbol{\alpha}]} c_{\boldsymbol{\delta}}\, \mathbf{x}'^{\boldsymbol{\delta}}$$

we have to transform $\mathbf{x}'^{\boldsymbol{\delta}}\, \partial^{\boldsymbol{\beta}'}_{\mathbf{x}'} \partial^{\boldsymbol{\beta}''}_{\mathbf{x}''} \eta$ with $[\boldsymbol{\delta}] = [\boldsymbol{\beta}'] + [\boldsymbol{\beta}'']$. Suppose that the variable x_j belongs to the first group $\mathbf{x}'$ with the corresponding multiindex $\mathbf{e}'_j \in \mathbf{Z}^n_{\geqslant 0}$ of norm 1. Similarly a variable of $\mathbf{x}''$ will be denoted by x_k with the corresponding multiindex $\mathbf{e}''_k \in \mathbf{Z}^{s-n}_{\geqslant 0}$ of norm 1. Since for any radial function η there holds the identity

$$x_j\, \partial_{x_k} \eta = x_k\, \partial_{x_j} \eta, \tag{3.12}$$

we have for $\boldsymbol{\beta}' = \mathbf{0}$

$$\mathbf{x}'^{\boldsymbol{\delta}}\, \partial^{\boldsymbol{\beta}''}_{\mathbf{x}''} \eta = \mathbf{x}''^{\boldsymbol{\beta}''}\, \partial^{\boldsymbol{\delta}}_{\mathbf{x}'} \eta. \tag{3.13}$$

Moreover, as long as $\boldsymbol{\delta} - \mathbf{e}'_j \in \mathbf{Z}^n_{\geqslant 0}$ and $\boldsymbol{\beta}'' - \mathbf{e}''_k \in \mathbf{Z}^{s-n}_{\geqslant 0}$, we obtain from (3.12) that

$$\begin{aligned}
\mathbf{x}'^{\boldsymbol{\delta}}\, \partial^{\boldsymbol{\beta}'}_{\mathbf{x}'} \partial^{\boldsymbol{\beta}''}_{\mathbf{x}''} \eta &= \mathbf{x}'^{\boldsymbol{\delta}-\mathbf{e}'_j} x_j\, \partial^{\boldsymbol{\beta}'}_{\mathbf{x}'} \partial^{\boldsymbol{\beta}''}_{\mathbf{x}''} \eta \\
&= \mathbf{x}'^{\boldsymbol{\delta}-\mathbf{e}'_j}\, \partial^{\boldsymbol{\beta}'}_{\mathbf{x}'} \partial^{\boldsymbol{\beta}''-\mathbf{e}''_k}_{\mathbf{x}''} x_j\, \partial_{x_k} \eta - \mathbf{x}'^{\boldsymbol{\delta}-\mathbf{e}'_j}\, \partial^{\boldsymbol{\beta}'-\mathbf{e}'_j}_{\mathbf{x}'} \partial^{\boldsymbol{\beta}''}_{\mathbf{x}''} \eta \\
&= \mathbf{x}'^{\boldsymbol{\delta}-\mathbf{e}'_j}\, \partial^{\boldsymbol{\beta}'}_{\mathbf{x}'} \partial^{\boldsymbol{\beta}''-\mathbf{e}''_k}_{\mathbf{x}''} x_k\, \partial_{x_j} \eta - \mathbf{x}'^{\boldsymbol{\delta}-\mathbf{e}'_j}\, \partial^{\boldsymbol{\beta}'-\mathbf{e}'_j}_{\mathbf{x}'} \partial^{\boldsymbol{\beta}''}_{\mathbf{x}''} \eta \\
&= \mathbf{x}'^{\boldsymbol{\delta}-\mathbf{e}'_j} x_k\, \partial^{\boldsymbol{\beta}'+\mathbf{e}'_j}_{\mathbf{x}'} \partial^{\boldsymbol{\beta}''-\mathbf{e}''_k}_{\mathbf{x}''} \eta + \mathbf{x}'^{\boldsymbol{\delta}-\mathbf{e}'_j}\, \partial^{\boldsymbol{\beta}'+\mathbf{e}'_j}_{\mathbf{x}'} \partial^{\boldsymbol{\beta}''-2\mathbf{e}''_k}_{\mathbf{x}''} \eta \\
&\quad - \mathbf{x}'^{\boldsymbol{\delta}-\mathbf{e}'_j}\, \partial^{\boldsymbol{\beta}'-\mathbf{e}'_j}_{\mathbf{x}'} \partial^{\boldsymbol{\beta}''}_{\mathbf{x}''} \eta.
\end{aligned}$$

If the vectors $\boldsymbol{\beta}'' - 2\mathbf{e}''_k$ or $\boldsymbol{\beta}' - \mathbf{e}'_j$ have negative components then the corresponding terms of the right hand side are set to 0.

Obviously, repeating the last equations and using (3.13) we obtain finally expressions of the form

$$\mathbf{x}'^{\boldsymbol{\gamma}'}\mathbf{x}''^{\boldsymbol{\gamma}''}\,\partial_{\mathbf{x}'}^{\boldsymbol{\delta}}\eta, \qquad 1 \leqslant [\boldsymbol{\gamma}'] + [\boldsymbol{\gamma}''] = [\boldsymbol{\delta}] \leqslant [\boldsymbol{\beta}],$$

which, in view of $\mathbf{x} = A\mathbf{y}$ and $\mathbf{y} = B\mathbf{x}'$, completes the proof. ▮

Hence the proof of Theorem 3.1 follows from

LEMMA 3.6. *Let* $A: \mathbf{R}^n \to \mathbf{R}^s$, $s > n$, *be a linear mapping of rank* n *and* η *is a smooth radial function in* $\mathbf{R}^s$. *Using the notations of Lemma* 3.5 *there holds*

$$\mathscr{F}\tilde{\eta}_{A,\boldsymbol{\beta}}(\boldsymbol{\lambda}) = \sum_{[\boldsymbol{\alpha}]=[\boldsymbol{\beta}]} \sum_{[\boldsymbol{\delta}]=1}^{[\boldsymbol{\beta}]} \sum_{[\boldsymbol{\gamma}]=[\boldsymbol{\delta}]} d_{\boldsymbol{\gamma},\boldsymbol{\delta}} \frac{(-1)^{[\boldsymbol{\beta}]+[\boldsymbol{\gamma}]}\,[\boldsymbol{\beta}]!}{(2\pi i)^{[\boldsymbol{\beta}]}\,\boldsymbol{\alpha}!} \times \partial^{\boldsymbol{\alpha}+\boldsymbol{\gamma}}\,(\boldsymbol{\lambda}^{\boldsymbol{\delta}}\mathscr{F}\eta((A^*A)^{-1/2}\,\boldsymbol{\lambda})).$$

In particular, under the moment condition (1.2)

$$\partial^{\boldsymbol{\alpha}}\mathscr{F}\tilde{\eta}_{A,\boldsymbol{\beta}}(\mathbf{0}) = \begin{cases} 1, & [\boldsymbol{\alpha}] = [\boldsymbol{\beta}] = 0, \\ 0, & 1 \leqslant [\boldsymbol{\beta}] + [\boldsymbol{\alpha}] \leqslant N-1. \end{cases}$$

4. QUASI-INTERPOLANT OF GENERAL FORM

Now we apply the estimate (3.5) for (3.9) to the quasi-interpolation formula (1.1) containing the numbers $V_{\mathbf{m}}$, which should be determined by using only the centers around $\mathbf{x}_{\mathbf{m}}$ rather than the in general unknown parametrization $\boldsymbol{\phi}$. Of course $V_{\mathbf{m}}$ will be an approximation of $h\,|\boldsymbol{\phi}'(h\mathbf{m})|^{1/n}$.

THEOREM 4.1. *Under the conditions of Theorem* 3.1 *the quasi-interpolant*

$$u_h(\mathbf{x}) = \mathscr{D}^{-n/2} \sum_{\mathbf{x}_{\mathbf{m}} \in \boldsymbol{\phi}(B(\mathbf{y}_0,\,\kappa))} u(\mathbf{x}_{\mathbf{m}})\,\eta\left(\frac{\mathbf{x}-\mathbf{x}_{\mathbf{m}}}{\sqrt{\mathscr{D}}\,V_{\mathbf{m}}}\right), \qquad \mathbf{x} \in \Omega,$$

approximates sufficiently smooth functions u *with estimate* (3.5) *if the numbers* $V_{\mathbf{m}}$ *satisfy*

$$|(V_{\mathbf{m}})^n - h^n\,|\boldsymbol{\phi}'(h\mathbf{m})|\,| \leqslant c h^{N+n}.$$

Proof. Evidently,

$$\sum_{\mathbf{x_m} \in \xi(B(\mathbf{y}_0,\,\kappa))} u(\mathbf{x_m}) \left(\eta \left(\frac{\mathbf{x} - \mathbf{x_m}}{\sqrt{\mathscr{D}}\, h\, |\boldsymbol{\phi}'(h\mathbf{m})| \pi 1/n} \right) - \eta \left(\frac{\mathbf{x} - \mathbf{x_m}}{\sqrt{\mathscr{D}}\, V_{\mathbf{m}}} \right) \right)$$

$$= \sum_{[\boldsymbol{\alpha}] = 1} \; \sum_{\mathbf{x_m} \in \xi(B(\mathbf{y}_0,\,\kappa))} u(\mathbf{x_m}) \frac{V_{\mathbf{m}} - h\, |\boldsymbol{\phi}'(h\mathbf{m})|^{1/n}}{V_{\mathbf{m}}}$$

$$\times \eta_{\boldsymbol{\alpha}} \left(\frac{\mathbf{x} - \mathbf{x_m}}{\sqrt{\mathscr{D}}\, h\, |\boldsymbol{\phi}'(h\mathbf{m})|^{1/n}} \right) + R(\mathbf{x}),$$

where we use the notation $\eta_{\boldsymbol{\alpha}}(\mathbf{x}) = \mathbf{x}^{\boldsymbol{\alpha}} \partial^{\boldsymbol{\alpha}} \eta(\mathbf{x})$ and the function $R(\mathbf{x})$ is uniformly bounded by

$$|R(\mathbf{x})| \leqslant c \left(\frac{V_{\mathbf{m}} - h\, |\boldsymbol{\phi}'(h\mathbf{m})|^{1/n}}{V_{\mathbf{m}}} \right)^2.$$

Since

$$\int_{\mathbf{R}^n} \eta_{\boldsymbol{\alpha}}(\mathbf{y})\, d\mathbf{y} = -1, \qquad [\boldsymbol{\alpha}] = 1,$$

it is clear from Theorem 3.1 and Lemma 3.1 that

$$\mathscr{D}^{-n/2} \sum_{\mathbf{x_m} \in \xi(B(\mathbf{y}_0,\,\kappa))} u(\mathbf{x_m}) \left(\eta \left(\frac{\mathbf{x} - \mathbf{x_m}}{\sqrt{\mathscr{D}}\, h\, |\boldsymbol{\phi}'(h\mathbf{m})|^{1/n}} \right) - \eta \left(\frac{\mathbf{x} - \mathbf{x_m}}{\sqrt{\mathscr{D}}\, V_{\mathbf{m}}} \right) \right)$$

is of the order $\mathcal{O}(h^N)$ if and only if

$$\left| \frac{V_{\mathbf{m}} - h\, |\boldsymbol{\phi}'(h\mathbf{m})|^{1/n}}{V_{\mathbf{m}}} \right| \leqslant c h^N$$

for all data points $\mathbf{x_m}$ in the neighbourhood $\xi(B(\mathbf{y}_0, \kappa))$ of $\mathbf{x}$. ∎

There exist different methods to find Nth order approximations of $|\boldsymbol{\phi}'(h\mathbf{m})|$. The simplest way is to replace the partial derivatives $\partial \varphi_j / \partial y_\ell$ by difference quotient approximations Δ_ℓ^j of the order h^N which involve only the jth coordinates $x_{\mathbf{k}}^j$ of the centers $\mathbf{x_k} = \boldsymbol{\phi}(h\mathbf{k})$, $\mathbf{k} \in \mathbf{Z}^n$, near $\mathbf{x_m}$. An example is given by

$$\Delta_\ell^j := \frac{2(x_{\mathbf{m}+\mathbf{e}_\ell}^j - x_{\mathbf{m}-\mathbf{e}_\ell}^j)}{3h} - \frac{x_{\mathbf{m}+2\mathbf{e}_\ell}^j - x_{\mathbf{m}-2\mathbf{e}_\ell}^j}{12h}$$

approximating $\partial\varphi_j/\partial y_\ell$ with the order h^4. Obviously the nth root of

$$|\|h\Delta_\ell^j\||, \qquad j=1, ..., s, \quad \ell=1, ..., n,$$

can be taken as $V_{\mathbf{m}}$ if $N=4$.

Another method of defining $V_{\mathbf{m}}$ which uses the measure of grid patches near $\mathbf{x_m}$ consists in the following. Consider a cube $Q\subset\mathbf{R}^n$ having all the corners at lattice points $\mathbf{k}\in\mathbf{Z}^n$ with $\mathbf{0}\in Q$. We denote its volume by $|Q|$ and introduce $Q_{\mathbf{m}}=h\mathbf{m}+hQ$. Then

$$\frac{1}{|Q|}\int_{Q_{\mathbf{m}}}|\boldsymbol{\phi}'(b\mathbf{y})|\,d\mathbf{y}=h^n\,|\boldsymbol{\phi}'(h\mathbf{m})|+\sum_{[\boldsymbol{\alpha}]=1}^{N-1}\frac{\partial^{\boldsymbol{\alpha}}\,|\boldsymbol{\phi}'(h\mathbf{m})|}{\boldsymbol{\alpha}!}\,\frac{h^{n+[\boldsymbol{\alpha}]}}{|Q|}$$
$$\times\int_Q \mathbf{y}^{\boldsymbol{\alpha}}\,d\mathbf{y}+\mathcal{O}(h^{N+n}). \tag{4.1}$$

Therefore, by choosing different cubes Q^j of the above mentioned type one can form a linear combination of the equalities (4.1) such that its sum does not contain terms with $h^{n+[\boldsymbol{\alpha}]}$, $1<[\boldsymbol{\alpha}]<N-1$. Thus, $h^n\,|\boldsymbol{\phi}'(h\mathbf{m})|$ can be approximated with the order $\mathcal{O}(h^{N+n})$ by linear combinations of the integrals

$$\int_{Q^j_{\mathbf{m}}}|\boldsymbol{\phi}'(\mathbf{y})|\,d\mathbf{y}=\int_{\boldsymbol{\phi}(Q^j_{\mathbf{m}})}d\mathbf{x}=|\boldsymbol{\phi}(Q^j_{\mathbf{m}})|$$

over the finite number of cubes $Q^j_{\mathbf{m}}=h\mathbf{m}+hQ^j$.

Consider the example of a surface Γ in $\mathbf{R}^3$. We choose the squares $Q^1=[-1,1]^2$ and Q^2 with corners at the points $(\pm 1, 0)$ and $(0, \pm 1)$. Then

$$\int_{Q^1_{\mathbf{m}}}|\boldsymbol{\phi}'(\mathbf{y})|\,d\mathbf{y}=4h^2\,|\boldsymbol{\phi}'(h\mathbf{m})|$$
$$+\frac{2h^4}{3}\,(\partial^{(2,0)}\,|\boldsymbol{\phi}'(h\mathbf{m})|+\partial^{(0,2)}\,|\boldsymbol{\phi}'(h\mathbf{m})|)+\mathcal{O}(h^6),$$

$$\int_{Q^2_{\mathbf{m}}}|\boldsymbol{\phi}'(\mathbf{y})|\,d\mathbf{y}=2h^2\,|\boldsymbol{\phi}'(h\mathbf{m})|$$
$$+\frac{h^4}{6}\,(\partial^{(2,0)}\,|\boldsymbol{\phi}'(h\mathbf{m})|+\partial^{(0,2)}\,|\boldsymbol{\phi}'(h\mathbf{m})|)+\mathcal{O}(h^6).$$

Consequently the quasi-interpolation formula (1.1) on Γ with

$$\eta(|\mathbf{x}|) = \pi^{-1} e^{-|\mathbf{x}|^2} \qquad \text{and} \qquad V_{\mathbf{m}} = \sqrt{|\phi(Q^1_{\mathbf{m}})|/2} \quad \text{or} \quad V_{\mathbf{m}} = \sqrt{|\phi(Q^2_{\mathbf{m}})|/2}$$

approximates with the order $\mathcal{O}(\mathcal{D}h^2)$ plus some saturation error. Similarly, approximate approximation with the order $\mathcal{O}(\mathcal{D}^2 h^4)$ on Γ can be obtained with the generating function

$$\eta(|\mathbf{x}|) = \pi^{-1}(2 - |\mathbf{x}|)\, e^{-|\mathbf{x}|^2} \qquad \text{and} \qquad V_{\mathbf{m}} = \sqrt{|\phi(Q^2_{\mathbf{m}})| - |\phi(Q^1_{\mathbf{m}})|/4}.$$

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